

Improvement of measurement accuracy in SU(1,1) interferometers

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Abstract

We consider an SU(1,1) interferometer employing four-wave mixers that is fed with two-mode states which are both coherent and intelligent states of the SU(1,1) Lie group. It is shown that the phase sensitivity of the interferometer can be essentially improved by using input states with a large photon-number difference between the modes.

The improvement of measurement accuracy in interferometers is of significant importance in modern experimental physics. Much work has been done on the reduction of the quantum noise in interferometers by using input light fields prepared in nonclassical photon states. It was pointed out by Caves [1] and Bondurant and Shapiro [2] that the quantum fluctuations can be diminished by feeding squeezed states of light into the interferometer. The interferometers considered in [1, 2] employ passive lossless devices, such as beam splitters. Yurke, McCall and Klauder [3] showed that such interferometers can be characterized by the SU(2) group. They also introduced a class of interferometers which employ active lossless devices, such as four-wave mixers, and are characterized by the SU(1,1) group. It was shown [3] that the use of squeezed light in SU(2) interferometers can yield a phase sensitivity $\Delta\phi \sim 1/N$ (where N is the total number of photons passing through the interferometer), while SU(1,1) interferometers can achieve a phase sensitivity of $1/N$ with only vacuum fluctuations entering the input ports.

In the present work we study the possibility to improve further the accuracy of SU(1,1) interferometers by using specially prepared states (other than vacuum). We apply the idea of Hillery and Mlodinow [4] who proposed to use intelligent states (IS) [5] for improving the phase sensitivity of interferometers. They analysed [4] the case of SU(2) IS. Since we discuss here interferometers characterized by SU(1,1), it is natural to use IS of this group [6, 7, 8]. There is a problem of generating IS since, in general, they are constructed by nonunitary operators [5, 6]. However, there are some IS which simultaneously are generalized coherent states (CS) [9, 10] of the corresponding Lie group, i.e., an intersection occurs between these two types of states [6]. This intersection is of special importance in physics because IS that also are CS can be created by Hamiltonians for which a given Lie group is the dynamical symmetry group. Recently we developed [7] a general group-theoretical approach to SU(1,1) IS by representing them in the corresponding coherent-state basis. This approach yields the most full characterization of the coherent-intelligent intersection. The above results will be used in the present work for analysing SU(1,1) interferometers fed with states which are both IS and CS of the SU(1,1) Lie group.

An SU(1,1) interferometer is described schematically in figure 1. Two light beams represented by mode annihilation operators a_1 and a_2 enter into the input ports of the first four-wave mixer FWM1. After leaving FWM1, beams accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then they enter

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the second four-wave mixer FWM2. The photons leaving the interferometer are counted by detectors D1 and D2.

For the analysis of such an interferometer it is convenient to consider the Hermitian operators

$$K_1 = \frac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2), \quad K_2 = \frac{1}{2i}(a_1^\dagger a_2^\dagger - a_1 a_2), \quad K_3 = \frac{1}{2}(a_1^\dagger a_1 + a_2 a_2^\dagger). \quad (1)$$

These operators form the two-mode boson realization of the $SU(1,1)$ Lie algebra:

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \quad (2)$$

It is also useful to introduce raising and lowering operators

$$K_+ = K_1 + iK_2 = a_1^\dagger a_2^\dagger, \quad K_- = K_1 - iK_2 = a_1 a_2. \quad (3)$$

The Casimir operator

$$K^2 = K_3^2 - K_1^2 - K_2^2 \quad (4)$$

for any unitary irreducible representation is the identity operator I times a number:

$$K^2 = k(k-1)I. \quad (5)$$

Thus a representation of $SU(1,1)$ is determined by a single number k that is called Bargmann index. For the discrete-series representations [11] the Bargmann index acquires discrete values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. By using the operators (1), one gets

$$K^2 = \frac{1}{4}(a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}. \quad (6)$$

The photon-number difference between the modes $n_0 = \langle a_1^\dagger a_1 - a_2^\dagger a_2 \rangle$ is a constant (chosen to be positive) for each irreducible representation and it is related to the Bargmann index via $k = \frac{1}{2}(n_0 + 1)$. The corresponding state space is spanned by the complete orthonormal basis $|k, n\rangle$ ($n = 0, 1, 2, \dots$) that can be expressed in terms of Fock states of two modes:

$$|k, n\rangle = |n + n_0\rangle_1 |n\rangle_2. \quad (7)$$

The actions of the interferometer elements on the vector $\mathbf{K} = (K_1, K_2, K_3)$ can be represented as Lorentz boosts and rotations in the (2+1)-dimensional space-time [3]. FWM1 acts on \mathbf{K} as a Lorentz boost with the transformation matrix

$$L(-\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \beta & -\sinh \beta \\ 0 & -\sinh \beta & \cosh \beta \end{pmatrix}. \quad (8)$$

The transformation matrix of FWM2 is $L(\beta)$, i.e., two four-wave mixers perform boosts in opposite directions. Phase shifters rotate \mathbf{K} about the 3rd axis by an angle $\phi = -(\phi_1 + \phi_2)$. The transformation matrix of this rotation is

$$R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

The overall transformation performed on \mathbf{K} is

$$\mathbf{K}_{\text{out}} = L(\beta)R(\phi)L(-\beta)\mathbf{K}. \quad (10)$$

The information on ϕ is inferred from the photon statistics of the output beams. One should measure the total number of photons in the two output modes, N_{out} , or, equivalently, the operator $K_{3\text{out}} = \frac{1}{2}(N_{\text{out}} + 1)$. The mean-square fluctuation in ϕ due to the photon statistics is given by [3]

$$(\Delta\phi)^2 = \frac{(\Delta K_{3\text{out}})^2}{|\partial\langle K_{3\text{out}}\rangle/\partial\phi|^2}. \quad (11)$$

From Eq. (10), we find

$$K_{3\text{out}} = (\sinh\beta \sin\phi)K_1 + \sinh\beta \cosh\beta(\cos\phi - 1)K_2 + (\cosh^2\beta - \sinh^2\beta \cos\phi)K_3. \quad (12)$$

If only vacuum fluctuations enter the input ports, then Eq. (11) with $K_{3\text{out}}$ of form (12) reduces to the known result [3]

$$(\Delta\phi)^2 = \frac{\sin^2\phi + \cosh^2\beta(1 - \cos\phi)^2}{\sin^2\phi \sinh^2\beta}. \quad (13)$$

For $\phi = 0$ the $(\Delta\phi)^2$ is minimized, $(\Delta\phi)^2 = 1/\sinh^2\beta$.

We would like to investigate a more general case when the interferometer is fed with an SU(1,1) intelligent state. The motivation for using IS is as follows. By putting $\phi = 0$, we can simplify Eq. (11) with $K_{3\text{out}}$ given by (12) to the form

$$(\Delta\phi)^2 = \frac{(\Delta K_3)^2}{\sinh^2\beta|\langle K_1\rangle|^2}. \quad (14)$$

The commutation relation $[K_2, K_3] = iK_1$ implies the uncertainty relation

$$(\Delta K_2)^2(\Delta K_3)^2 \geq \frac{1}{4}|\langle K_1\rangle|^2. \quad (15)$$

Therefore,

$$(\Delta\phi)^2 \geq \frac{1}{4\sinh^2\beta(\Delta K_2)^2}. \quad (16)$$

For IS an equality is achieved in the uncertainty relation. Such K_2 - K_3 IS with large values of ΔK_2 would allow us to measure small changes in ϕ . For these states Eq. (16) reads

$$(\Delta\phi)^2 = \frac{1}{4\sinh^2\beta(\Delta K_2)^2}. \quad (17)$$

The K_2 - K_3 IS $|\lambda\rangle_{23}$ are determined from the eigenvalue equation

$$(K_2 + i\gamma K_3)|\lambda\rangle_{23} = \lambda|\lambda\rangle_{23}, \quad (18)$$

where λ is a complex eigenvalue and γ is a real parameter given by

$$|\gamma| = \Delta K_2/\Delta K_3. \quad (19)$$

For $|\gamma| > 1$ IS are squeezed in K_3 and for $|\gamma| < 1$ IS are squeezed in K_2 .

In order to be able to create IS, we must choose states which lie in the intersection of the SU(1,1) intelligent and coherent states. The generalized SU(1,1) CS were introduced by Perelomov [9]:

$$\begin{aligned} |k, \zeta\rangle &= \exp(\xi K_+ - \xi^* K_-)|k, 0\rangle \\ &= (1 - |\zeta|^2)^k \exp(\zeta K_+)|k, 0\rangle \\ &= (1 - |\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{n!\Gamma(2k)} \right]^{1/2} \zeta^n |k, n\rangle. \end{aligned} \quad (20)$$

Here $\zeta = (\xi/|\xi|)\tanh|\xi|$, so $|\zeta| < 1$. In the case of the two-mode boson realization the SU(1,1) CS can be recognized as well-known two-mode squeezed states with ξ being a squeezing parameter [6]. Any intelligent state can be represented in the coherent-state basis [7]. By using this analytic representation, we can find that a K_2 - K_3 intelligent state is also coherent when its eigenvalue λ is [7]

$$\lambda = \pm ik\sqrt{\gamma^2 + 1}. \quad (21)$$

The corresponding coherent-state amplitude ζ is real:

$$\zeta = \frac{1}{\gamma \pm \sqrt{\gamma^2 + 1}}. \quad (22)$$

The condition $|\zeta| < 1$ is satisfied if $\gamma > 0$ for upper sign and $\gamma < 0$ for lower sign. Squeezing in K_3 ($|\gamma| > 1$) corresponds to values $|\zeta| < 0.414$.

By using the definition (20) of the SU(1,1) CS, one can easily calculate the variance of K_2 [6]

$$(\Delta K_2)^2 = \frac{k(1 + |\zeta|^4 - \zeta^2 - \zeta^{*2})}{2(1 - |\zeta|^2)^2}. \quad (23)$$

For a coherent-intelligent state, ζ is real, so we obtain $(\Delta K_2)^2 = k/2$. Other expectation values over $|k, \zeta\rangle$ are

$$(\Delta K_3)^2 = \frac{2k|\zeta|^2}{(1 - |\zeta|^2)^2}, \quad (24)$$

$$\langle K_1 \rangle = \frac{2k\text{Re } \zeta}{1 - |\zeta|^2}. \quad (25)$$

Then it is straightforward to check that an equality is achieved in the uncertainty relation (15) provided that expectation values are calculated over an SU(1,1) coherent state with real ζ . It is seen that the states that belong to the coherent-intelligent intersection lead to the best measurement accuracy among all the SU(1,1) CS. The mean-square fluctuation in ϕ given by Eq. (17) is, for the interferometer fed with an SU(1,1) coherent-intelligent state,

$$(\Delta\phi)^2 = \frac{1}{2k\sinh^2\beta}. \quad (26)$$

We see that the phase sensitivity is independent of the value of squeezing represented by ζ . It depends only on parameter β of the four-wave mixer and on the photon-number difference between the two input modes ($n_0 = 2k - 1$). Therefore, ζ can be taken to be zero, i.e., one can choose an input state with a fixed number of photons in the one mode and the vacuum in the other. The value of β is restricted by properties of available four-wave mixers. We see from Eq. (26) that for a given value of β the phase sensitivity of the SU(1,1) interferometers can be essentially improved by choosing input states with large values of the photon-number difference between the two modes. When $n_0 = 0$ (in particular, when the vacuum enters both input ports), the phase fluctuations come to the known value $(\Delta\phi)^2 = 1/\sinh^2\beta$.

It is usual to examine the interferometer efficiency by expressing the phase sensitivity $\Delta\phi$ in terms of the total number N of photons passing through the phase shifters. In the case of the interferometer considered here, N is the total number of photons emitted by FWM1:

$$N = 2\langle K'_3 \rangle - 1, \quad (27)$$

where $\mathbf{K}' = L(-\beta)\mathbf{K}$, so $K'_3 = (\cosh\beta)K_3 - (\sinh\beta)K_2$. By calculating the expectation value over a coherent-intelligent state, we obtain

$$N = 2k\frac{1 + \zeta^2}{1 - \zeta^2}\cosh\beta - 1. \quad (28)$$

Solving this equation for $\sinh^2\beta$ we finally find

$$(\Delta\phi)^2 = \frac{1}{2k} \left[\left(\frac{1-\zeta^2}{1+\zeta^2} \frac{N+1}{2k} \right)^2 - 1 \right]^{-1}. \quad (29)$$

We see that the phase sensitivity $\Delta\phi$ approaches $1/N$. The best interferometer efficiency is achieved for $\zeta = 0$ and $n_0 = 0$ ($k = 1/2$). Then one gets the result for the vacuum input¹ [3]:

$$(\Delta\phi)^2 = \frac{1}{N(N+2)}. \quad (30)$$

For a given N , $(\Delta\phi)^2$ is optimized by taking the vacuum in both input modes. However, for a given value of β (dictated by practical considerations), the phase sensitivity is improved by choosing input modes with a large photon-number difference between them.

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Figure captions

Figure 1: An SU(1,1) interferometer. Two light modes a_1 and a_2 are mixed by four-wave mixer FWM1, accumulate phase shifts ϕ_1 and ϕ_2 , respectively, and then they are again mixed by four-wave mixer FWM2. The photons in output modes are counted by detectors D1 and D2.

¹Please note a minor difference between this result and equation (9.31) of Ref. [3] where ‘ -2 ’ is erroneously printed instead of ‘ $+2$ ’.